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VIBRATION OF PRETWISTED CANTILEVER SHALLOW CONICAL SHELLS

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Abstract—This paper presents a mathematical model to investigate the effects of initial twist on the vibratory characteristics of cantilever shallow conical shells. The energy functional is minimized according to the Ritz procedure to arrive at the governing eigenvalue equation. A set of orthogonally generated two-dimensional polynomials associated with a basic function, which accounts for the boundary expressions and constraints, is introduced to approximate the in-plane and transverse displacement amplitude functions. The complete procedure has been automated to compute the vibration frequencies and mode shapes for exemplary problems. In the numerical experiments, the convergence of eigenvalues is confirmed by increasing the degrees of polynomials employed in the admissible shape functions. To enhance the existing literature, a set of first known frequency parameters is presented. The paper highlights the important effects of angle of twist on the vibration frequencies and mode shapes of conical shells. The fundamental physical frequency ω decreases monotonically for a longer conical shell. The result shows that an increase in the angle of twist does not ensure higher torsional stiffness for a conical shell, which is in contradiction with previous observation for a pretwisted beam or plate. The symmetry of modes is absent when the angle of twist is non-zero.

1. INTRODUCTION

Extensive practical uses of cantilever pretwisted shallow conical shells can be found in various engineering disciplines ranging from aerospace and marine industries to civil and structural applications. The vibratory characteristics are thus of critical influence to the performance and safety of these structures. In the early days, they had been normally modelled as a pretwisted cantilevered beam (Carnegie, 1959). This treatment is only justifiable provided that the structures are slender or only lower vibration modes are required. The effects of initial twist on the torsional rigidity of beams have been investigated by Chu (1951), Rosen (1980) and Shield (1982). They showed that the torsional rigidity of beams increases for increasing angle of twist. There exists also numerical models and experimental investigations of turbomachinery blades using cantilever pretwisted plates such as those reported by Leissa *et al.* (1984), MacBain *et al.* (1985), Kielb *et al.* (1985) and Liew and Lim (1994). These studies omitted the variable surface curvature of the blade. The vibration study of twisted thin cylindrical shells by Tsuiji and Sueoka (1990) has reported that the frequencies of torsional modes do not increase monotonically as the angle of twist increases.

Innumerable references dealing with the vibration analysis of closed conical shells can be found (Leissa, 1973; Chang, 1981). Despite its practical importance, the vibration of open conical shells has received relatively little attention and none of them, to the authors' knowledge, accounted for pretwisted shallow conical shells. Results on the vibrations of untwisted shallow conical shells have been reported by Lim and Liew (1994a).

To fill this apparent void, the present study employs an energy approach to examine the effects of initial twist upon the vibration characteristics of pretwisted shallow conical shells. The energy functional is minimized in accordance with the Ritz procedure. A set of orthogonally generated two-dimensional polynomials (p-2) associated with a basic function (b) is introduced to represent the admissible in-plane and transverse displacement amplitude functions. These pb-2 shape functions ensure the automatic satisfaction of the geometric boundary conditions because the piecewise boundary expressions and their respective constraints are formulated as intrinsic components of the basic functions. Consequently,

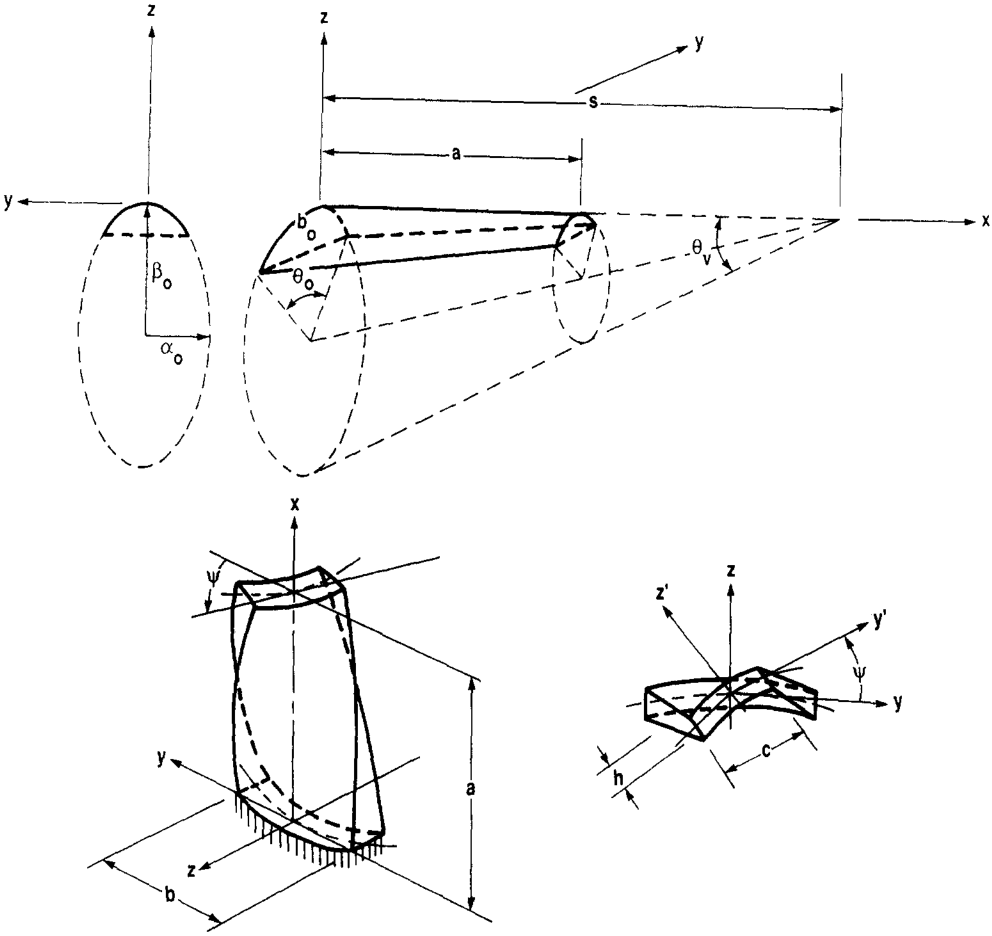

Fig. 1. Geometry of the (a) untwisted shallow conical shell with trapezoidal planform; (b) of the pretwisted shallow conical shell.

this new approach is highly versatile in accommodating various boundary conditions. Mathematical difficulties in complex shell geometry such as trapezoidal or triangular planform can also be easily overcome. Numerical formulation and computational implementation are greatly simplified since no mesh generation is needed. Furthermore, only relatively small amounts of computational memory and execution time are required and yet accurate results can be achieved because only a single global element is involved.

In this study, the numerical convergence of eigenvalues is carefully verified through a convergence study. To enhance the existing literature, a set of first known non-dimensional frequency parameter covering wide ranges of angle of twist and a/s ratio [see Fig. 1(a)] is presented. The effects of initial twist on the vibration frequency will be discussed in depth and also illustrated by means of mode shape displacement plots.

2. FORMULATION OF ENERGY EQUATIONS

Consider a homogeneous, isotropic, thin shallow conical shell with length a, reference width b_0, thickness h, vertex angle θ_v and base subtended angle of cone θ_0 as depicted in Fig. 1(a). Since the conical shell is shallow, it may be assumed that the cross section in Fig. 1(a) is elliptical. The component of radius of curvature in the chordwise direction $R_y(x, y)$ is a parameter varying both in the x- and y-directions. The variation in the x-direction is linear. There is no curvature along the spanwise direction ($R_x = \infty$). The cantilever shell, clamped along $x = 0$, is pretwisted with radius of twist R_{xy} as shown in Fig. 1(b). The radius of twist is related to the angle of twist ψ by

$$\tan\psi = -\frac{a}{R_{xy}}. \tag{1}$$

The displacements are resolved into three orthogonal components u, v and w with respect to the midsurface of the shell with u along the x-axis, v tangential to the midsurface and w normal to it.

The total strain energy, $\mathscr{U}$, is given by

$$\mathscr{U} = \mathscr{U}_s + \mathscr{U}_b, \tag{2}$$

where $\mathscr{U}_s$ is the membrane strain energy due solely to the stretching effects of the midsurface and $\mathscr{U}_b$ is the bending strain energy of the shell.

The strain energy components can be expressed as

$$\mathscr{U}_s = \frac{6D}{h^2}\iint_A \left[(\varepsilon_x+\varepsilon_y)^2 - 2(1-\nu)\left(\varepsilon_x\varepsilon_y - \frac{1}{4}\gamma_{xy}^2\right)\right]\mathrm{d}x\,\mathrm{d}y \tag{3}$$

and

$$\mathscr{U}_b = \frac{D}{2}\iint_A \left\{(\Delta w)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\}\mathrm{d}x\,\mathrm{d}y, \tag{4}$$

where the flexural rigidity $D = Eh^3/12(1-\nu^2)$, E is Young's modulus, ν is the Poisson ratio and Δ is the Laplacian operator defined as $(\partial^2/\partial x^2 + \partial^2/\partial y^2)$. The double integration above is performed throughout the projected trapezoidal planform of conical shell A.

The strains of the membrane can be expressed in terms of the displacements as

$$\varepsilon_x = \frac{\partial u}{\partial x} \tag{5a}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{w}{R_y(x,y)} \tag{5b}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{2w}{R_{xy}}. \tag{5c}$$

The kinetic energy is given by

$$\mathscr{T} = \frac{\rho h}{2}\iint_A \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right]\mathrm{d}x\,\mathrm{d}y, \tag{6}$$

where ρ is the mass density per unit volume.

Assuming the free vibration amplitude to be small, the displacement functions take the following forms,

$$u(x,y,t) = U(x,y)\sin\omega t \tag{7a}$$

$$v(x,y,t) = V(x,y)\sin\omega t \tag{7b}$$

$$w(x,y,t) = W(x,y)\sin\omega t, \tag{7c}$$

where ω denotes the angular frequency of vibration. Using eqns (5a–c) and (7a–c), the strain energy and kinetic energy components [eqns (3), (4) and (6)] can be simplified to the following expressions:

$$(\mathscr{U}_s)_{max} = \frac{6D}{h^2}\iint_A \left\{\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{W}{R_y(x,y)}\right)^2 + 2\frac{W}{R_y(x,y)}\frac{\partial V}{\partial y}\right.$$

$$+2v\frac{\partial U}{\partial x}\left(\frac{\partial V}{\partial y}+\frac{W}{R_y(x,y)}\right)+\frac{1-v}{2}\left[\left(\frac{\partial V}{\partial x}\right)^2+2\frac{\partial V}{\partial x}\frac{\partial U}{\partial y}+\left(\frac{\partial U}{\partial y}\right)^2\right.$$

$$\left.\left.+\frac{4W}{R_{xy}}\left(\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}\right)+\frac{4W^2}{R_{xy}^2}\right]\right\}\mathrm{d}x\,\mathrm{d}y \tag{8a}$$

$$(\mathscr{U}_b)_{max} = \frac{D}{2}\iint_A \left\{(\Delta W)^2 - 2(1-v)\left[\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} - \left(\frac{\partial^2 W}{\partial x \partial y}\right)^2\right]\right\}\mathrm{d}x\,\mathrm{d}y \tag{8b}$$

$$\mathscr{T}_{max} = \frac{\rho h \omega^2}{2}\iint_A (U^2+V^2+W^2)\,\mathrm{d}x\,\mathrm{d}y, \tag{8c}$$

where $(\mathscr{U}_s)_{max}$, $(\mathscr{U}_b)_{max}$ and $\mathscr{T}_{max}$ are the respective maximum stretching strain energy, maximum bending strain energy and maximum kinetic energy in a vibratory circle.

3. FORMULATION OF EIGENVALUE EQUATIONS

Introducing the non-dimensional coordinate system as follows:

$$\xi = \frac{x}{a}, \quad \eta = \frac{y}{b_0}, \tag{9a, 9b}$$

where a and b_0 are the length and reference width of the shell planform as shown in Fig. 1(a). The displacement amplitude functions $U(\xi,\eta)$, $V(\xi,\eta)$ and $W(\xi,\eta)$ can then be approximated by a set of two-dimensional polynomials of the form

$$U(\xi,\eta) = \sum_{i=1}^{m} C_{ui}\phi_{ui}(\xi,\eta) \tag{10a}$$

$$V(\xi,\eta) = \sum_{i=1}^{m} C_{vi}\phi_{vi}(\xi,\eta) \tag{10b}$$

$$W(\xi,\eta) = \sum_{i=1}^{m} C_{wi}\phi_{wi}(\xi,\eta), \tag{10c}$$

in which C_{ui}, C_{vi} and C_{wi} are the unknown coefficients and ϕ_{ui}, ϕ_{vi} and ϕ_{wi} are the corresponding *pb*-2 shape functions to be discussed in due course. The varying radius of curvature can also be expressed in terms of this coordinate system,

$$R_y(\xi,\eta) = \frac{\beta_0}{f(\xi,\eta)}, \tag{11}$$

where β_0 is the reference major radius as shown in Fig. 1(a). The function $f(\xi,\eta)$ can be derived from the geometry of conical shell and is expressed as follows:

$$f(\xi,\eta) = \frac{\beta_0}{s}\frac{s}{R_y(\xi,\eta)}, \tag{12}$$

where

$$\frac{\beta_0}{s} = \tan\frac{\theta_v}{2} \tag{13a}$$

and

$$\frac{R_y(\xi,\eta)}{s} = \left(\frac{\alpha}{s}\right)^2\left(\frac{\beta}{s}\right)^2\left\{\left(\frac{s}{\beta}\right)^2 + \eta^2\left(\frac{b_0}{s}\right)^2\left(\frac{s}{\alpha}\right)^2\left[\left(\frac{s}{\alpha}\right)^2 - \left(\frac{s}{\beta}\right)^2\right]\right\}^{3/2}, \tag{13b}$$

in which

$$\frac{\alpha}{s} = \frac{\frac{b}{s}\frac{\beta}{s}\tan\frac{\theta_0}{2}}{\sqrt{4\left(\frac{\beta}{s}\right)^2\tan^2\frac{\theta_0}{2} - \left(\frac{b}{s}\right)^2}} \tag{14a}$$

$$\frac{\beta}{s} = \tan\frac{\theta_v}{2}\left[1 - \left(\frac{a}{s}\right)\xi\right] \tag{14b}$$

$$\frac{b_0}{s} = 2\sin\frac{\theta_v}{2}\sqrt{\frac{\tan^2\theta_0/2}{\cos^2\theta_v/2 + \tan^2\theta_0/2}} \tag{14c}$$

and

$$\frac{b}{s} = \frac{b_0}{s}\left[1 - \left(\frac{a}{s}\right)\xi\right]. \tag{14d}$$

Let Π be the energy functional given by

$$\Pi = \mathscr{U}_{\max} - \mathscr{T}_{\max}, \tag{15}$$

where $\mathscr{U}_{\max}$ is the sum of eqns (8a,b). This energy functional is then minimized with respect to the coefficients according to the Ritz procedure,

$$\frac{\partial \Pi}{\partial C_{\alpha i}} = 0; \quad \alpha = u, v \text{ and } w, \tag{16}$$

which leads to the governing eigenvalue equation,

$$(12\mathbf{K} - \lambda^2\mathbf{M})\{\mathbf{C}\} = \{\mathbf{0}\}, \tag{17}$$

where **K** is the stiffness matrix and **M** is the mass matrix expressed as follows:

$$\mathbf{K} = \begin{bmatrix} [K_{uu}] & [K_{uv}] & [K_{uw}] \\ & [K_{vv}] & [K_{vw}] \\ \text{sym} & & [K_{ww}] \end{bmatrix} \tag{18a}$$

$$\mathbf{M} = \begin{bmatrix} [M_{uu}] & [0] & [0] \\ & [M_{vv}] & [0] \\ \text{sym} & & [M_{ww}] \end{bmatrix} \tag{18b}$$

and the vector of unknown coefficients is

$$\{\mathbf{C}\} = \begin{Bmatrix} \{C_u\} \\ \{C_v\} \\ \{C_w\} \end{Bmatrix}. \tag{18c}$$

The elements in the stiffness and mass matrices are given by

$$K_{uuij} = \left(\frac{b_0}{h}\right)^2 \mathscr{I}^{1010}_{uiuj} + \left(\frac{1-v}{2}\right)\left(\frac{a}{h}\right)^2 \mathscr{I}^{0101}_{uiuj} \tag{19a}$$

$$K_{uvij} = v\left(\frac{a}{h}\right)\left(\frac{b_0}{h}\right)\mathscr{I}^{1001}_{uivj} + \left(\frac{1-v}{2}\right)\left(\frac{a}{h}\right)\left(\frac{b_0}{h}\right)\mathscr{I}^{0110}_{uivj} \tag{19b}$$

$$K_{uwij} = \left(\frac{a}{h}\right)\left(\frac{b_0}{h}\right)\left[\left(\frac{vb_0}{\beta_0}\right)\mathscr{I}^{1000}_{uiwj} + (1-v)\left(\frac{a}{R_{xy}}\right)\mathscr{I}^{0100}_{uiwj}\right] \tag{19c}$$

$$K_{vvij} = \left(\frac{a}{h}\right)^2 \mathscr{I}^{0101}_{vivj} + \left(\frac{1-v}{2}\right)\left(\frac{b_0}{h}\right)^2 \mathscr{I}^{1010}_{vivj} \tag{19d}$$

$$K_{vwij} = \left(\frac{a}{h}\right)\left(\frac{b_0}{h}\right)\left[\left(\frac{va}{\beta_0}\right)\mathscr{I}^{0100}_{viwj} + (1-v)\left(\frac{b_0}{R_{xy}}\right)\mathscr{I}^{1000}_{viwj}\right] \tag{19e}$$

$$K_{wwij} = \left(\frac{a}{h}\right)^2\left[\left(\frac{b_0}{\beta_0}\right)^2 + 2(1-v)\left(\frac{b_0}{R_{xy}}\right)^2\right]\mathscr{I}^{0000}_{wiwj}$$
$$+\frac{1}{12}\left[\left(\frac{b_0}{a}\right)^2 \mathscr{I}^{2020}_{wiwj} + \left(\frac{a}{b_0}\right)^2 \mathscr{I}^{0202}_{wiwj} + v\left(\mathscr{I}^{0220}_{wiwj} + \mathscr{I}^{2002}_{wiwj}\right) + 2(1-v)\mathscr{I}^{1111}_{wiwj}\right] \tag{19f}$$

$$M_{uuij} = \mathscr{I}^{0000}_{uiuj}; \quad M_{vvij} = \mathscr{I}^{0000}_{vivj} \tag{19g, 19h}$$

and

$$M_{wwij} = \mathscr{I}^{0000}_{wiwj}; \quad \lambda = \omega a b_0 \sqrt{\frac{\rho h}{D}}, \tag{19i, 19j}$$

in which

$$\mathscr{I}^{defg}_{uiuj} = \iint_A \frac{\partial^{d+e}\phi_{ui}(\xi,\eta)}{\partial\xi^d\partial\eta^e}\frac{\partial^{f+g}\phi_{uj}(\xi,\eta)}{\partial\xi^f\partial\eta^g}\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{20a}$$

$$\mathscr{I}^{defg}_{uivj} = \iint_A \frac{\partial^{d+e}\phi_{ui}(\xi,\eta)}{\partial\xi^d\partial\eta^e}\frac{\partial^{f+g}\phi_{vj}(\xi,\eta)}{\partial\xi^f\partial\eta^g}\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{20b}$$

$$\mathscr{I}^{defg}_{uiwj} = \iint_A f(\xi,\eta)\frac{\partial^{d+e}\phi_{ui}(\xi,\eta)}{\partial\xi^d\partial\eta^e}\frac{\partial^{f+g}\phi_{wj}(\xi,\eta)}{\partial\xi^f\partial\eta^g}\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{20c}$$

$$\mathscr{I}^{defg}_{vivj} = \iint_A \frac{\partial^{d+e}\phi_{vi}(\xi,\eta)}{\partial\xi^d\partial\eta^e}\frac{\partial^{f+g}\phi_{vj}(\xi,\eta)}{\partial\xi^f\partial\eta^g}\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{20d}$$

$$\mathscr{I}^{defg}_{viwj} = \iint_A f(\xi,\eta)\frac{\partial^{d+e}\phi_{vi}(\xi,\eta)}{\partial\xi^d\partial\eta^e}\frac{\partial^{f+g}\phi_{wj}(\xi,\eta)}{\partial\xi^f\partial\eta^g}\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{20e}$$

$$\mathscr{I}_{wiwj}^{defg} = \iint_A f(\xi,\eta)\frac{\partial^{d+e}\phi_{wi}(\xi,\eta)}{\partial\xi^d\partial\eta^e}\frac{\partial^{f+g}\phi_{wj}(\xi,\eta)}{\partial\xi^f\partial\eta^g}\,d\xi\,d\eta \tag{20f}$$

$$\mathscr{I}_{wiwj}^{defg} = \iint_A \frac{\partial^{d+e}\phi_{wi}(\xi,\eta)}{\partial\xi^d\partial\eta^e}\frac{\partial^{f+g}\phi_{wj}(\xi,\eta)}{\partial\xi^f\partial\eta^g}\,d\xi\,d\eta, \tag{20g}$$

where $i, j = 1, 2 \ldots m$ and m is determined by the degree of polynomial employed in the shape functions. The double integrations above are symmetric where

$$\mathscr{I}_{\alpha i\beta j}^{defg} = \mathscr{I}_{\beta j\alpha i}^{fgde}, \quad \mathscr{I}_{\alpha i\beta j}^{defg} = \mathscr{I}_{\beta j\alpha i}^{fgde}. \tag{21a, 21b}$$

4. ADMISSIBLE DISPLACEMENT FUNCTIONS

The admissible *pb*-2 shape functions $\phi_{\alpha i}$ ($\alpha = u$, v and w) are the product of sets of orthogonally generated complete two-dimensional polynomials and a kinematically oriented basic function expressed as follows:

$$\phi_{\alpha i}(\xi,\eta) = f_i(\xi,\eta)\phi_{\alpha 1} - \sum_{j=1}^{i-1}\Xi_{\alpha ij}\phi_{\alpha j}, \tag{22}$$

where

$$\Xi_{\alpha ij} = \frac{{}_1\Delta_{\alpha ij}}{{}_2\Delta_{\alpha j}} \tag{23a}$$

$${}_1\Delta_{\alpha ij} = \iint_A f_i(\xi,\eta)\phi_{\alpha 1}\phi_{\alpha j}\,d\xi\,d\eta \tag{23b}$$

$${}_2\Delta_{\alpha j} = \iint_A (\phi_{\alpha j})^2\,d\xi\,d\eta \tag{23c}$$

$$\alpha = u, v \text{ and } w.$$

The inner product of any two different terms in the series satisfies the orthogonality condition

$$\iint_A \phi_{\alpha i}(\xi,\eta)\phi_{\alpha j}(\xi,\eta)\,d\xi\,d\eta = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}. \tag{24}$$

The use of these two-dimensional orthogonal polynomials to approximate the transverse deflection (W) of an annular sector plate has been discussed in detail by Liew and Lam (1993). It has also been further extended to account for in-plane (U and V) and transverse (W) deflections by Lim and Liew (1994a, b). In the present analysis, the similar shape functions are used to approximate the in-plane and transverse deflections of a conical shell.

The basic function, $\phi_{\alpha 1}$, defined as the product of the equations of the continuous piecewise boundary geometric expressions each of which is raised to a basic power depending on the types of boundary constraints imposed on the shell, i.e.

$$\phi_{\alpha 1}(\xi,\eta) = \prod_{i=1}^{n_e}[\Gamma_i(\xi,\eta)]^{\gamma_{\alpha i}} \tag{25}$$

$$\alpha = u, v \text{ and } w,$$

where n_e is the number of supporting edges; Γ_i is the boundary expression of the ith supporting edge; $\gamma_{\alpha i}$, ($\alpha = u$, v and w) are the basic powers. The powers for the transverse boundary conditions ($\alpha = w$) are 0, 1 or 2 corresponding to a free, simply supported or clamped edge. For the in-plane boundary conditions ($\alpha = u$ and v), the power is 0 for a completely free edge and 1 for either a simply supported or clamped edge. This implies that the in-plane deflection gradients, $\partial u/\partial n_r \neq 0$, $\partial u/\partial n_s \neq 0$, $\partial v/\partial n_r \neq 0$ and $\partial v/\partial n_s \neq 0$, where n_r and n_s are directions normal and tangential to the shell peripheries.

The Ritz method requires an admissible function which satisfies the geometric boundary conditions. The imposition of the powers to the basic functions results in a class of kinematically oriented *pb*-2 shape functions consistent with the following geometric boundary conditions:

$$U|_{\xi=0} = 0; \quad V|_{\xi=0} = 0; \quad W|_{\xi=0} = 0 \qquad (26a, 26b, 26c)$$

and

$$\left.\frac{\partial W}{\partial \xi}\right|_{\xi=0} = 0. \qquad (26d)$$

For illustrative purposes, the basic functions for the cantilevered conical shell is

$$\phi_{u1} = \xi; \quad \phi_{v1} = \xi; \quad \phi_{w1} = \xi^2 \qquad (27a, 27b, 27c)$$

which satisfy the above geometric boundary conditions.

The two-dimensional polynomials $\Sigma_{i=1}^{m} f_i(\xi, \eta)$, can be expressed as

$$\sum_{i=1}^{m} f_i(\xi, \eta) = \sum_{q=0}^{p} \sum_{i=0}^{q} \xi^{q-i}\eta^{i} \qquad (28)$$

with m and p related by

$$m = \frac{(p+1)(p+2)}{2}, \qquad (29)$$

where p is the degree of the complete set of two-dimensional polynomials employed.

5. RESULTS AND DISCUSSION

Having derived the eigenvalue equation and method of solution, the algorithm is implemented to obtain solutions for vibration frequencies of pretwisted cantilever shallow conical shells. Since λ is a function of a as expressed in eqn (19j), another non-dimensional frequency parameter $\lambda' = \omega b_0^2\sqrt{\rho h/D}$, or equivalently $\lambda' = \lambda e_1/e_2$ where $e_1 = b_0/s$ is a constant which can be determined using eqn (14c) and $e_2 = a/s$, is introduced here. The physical frequency ω is proportional to λ' which is used in all the subsequent tables and figures.

Table 1. Convergence of $\lambda' = \omega b_0^2\sqrt{(\rho h/D)}$ for the pretwisted shallow conical shell with $\nu = 0.3$, $a/s = 0.5$, $s/h = 1000$, $\theta_v = 15°$, $\theta_0 = 30°$ and $\psi = 45°$

p			Mode sequence number							
u	v	w	1	2	3	4	5	6	7	8
10	10	10	0.23651	1.0297	2.5911	2.7863	4.9699	5.9293	6.4859	8.2920
10	10	11	0.23643	1.0292	2.5911	2.7854	4.9677	5.9279	6.4859	8.2882
10	10	12	0.23635	1.0290	2.5909	2.7844	4.9669	5.9270	6.4859	8.2839
10	10	13	0.23632	1.0287	2.5909	2.7839	4.9660	5.9259	6.4856	8.2822
10	10	14	0.23628	1.0287	2.5909	2.7835	4.9657	5.9252	6.4855	8.2801
10	10	15	0.23627	1.0286	2.5907	2.7833	4.9655	5.9246	6.4853	8.2794
10	11	15	0.23624	1.0285	2.5906	2.7829	4.9651	5.9236	6.4844	8.2672
10	12	15	0.23623	1.0284	2.5906	2.7827	4.9649	5.9229	6.4840	8.2636
10	13	15	0.23623	1.0284	2.5905	2.7827	4.9647	5.9226	6.4837	8.2627
10	14	15	0.23622	1.0284	2.5905	2.7825	4.9646	5.9225	6.4836	8.2626
10	15	15	0.23622	1.0284	2.5903	2.7825	4.9646	5.9223	6.4834	8.2623
11	15	15	0.23619	1.0282	2.5903	2.7821	4.9630	5.9214	6.4827	8.2423
12	15	15	0.23616	1.0281	2.5903	2.7817	4.9627	5.9212	6.4827	8.2322
13	15	15	0.23616	1.0280	2.5903	2.7816	4.9624	5.9210	6.4826	8.2318
14	15	15	0.23615	1.0280	2.5902	2.7814	4.9623	5.9208	6.4826	8.2311
15	15	15	0.23615	1.0280	2.5902	2.7814	4.9621	5.9208	6.4826	8.2308

The convergence of eigenvalues is shown in Table 1. Monotonic downward convergence is demonstrated signifying a distinct nature of Ritz procedure which always overestimates the structural stiffness of the conical shell. However, accuracy of results can be ensured by employing satisfactory degrees of polynomials for the in-plane and transverse shape functions. From Table 1, it is shown that degrees 15, 15 and 15 for u, v and w are sufficient to furnish the satisfactory convergent eigenvalues.

A set of non-dimensional frequency parameters covering wide ranges of a/s is presented in Table 2. The Poisson ratio is fixed at 0.3, thickness ratio s/h at 1000.0 while the vertex

Table 2. Frequency parameter $\lambda' = \omega b_0^2\sqrt{(\rho h/D)}$ for the pretwisted shallow conical shell with $\nu = 0.3$, $s/h = 1000$, $\theta_v = 15°$ and $\theta_0 = 30°$

ψ	$\frac{a}{s}$	Mode sequence number 1	2	3	4	5	6	7	8
0°	0.2	2.1021	3.0890	9.2965	10.134	17.255	22.194	25.370	27.312
	0.3	1.0274	2.1768	4.7404	6.0640	10.642	11.502	11.931	16.030
	0.4	0.63879	1.7850	2.6925	4.6908	6.5539	6.9417	7.9776	11.662
	0.5	0.45689	1.5872	1.7267	3.9728	4.1078	4.5848	6.6226	7.5149
	0.6	0.35997	1.2037	1.4840	2.7622	3.2709	3.6000	5.0253	5.8618
	0.7	0.30608	0.89631	1.4323	1.9593	2.5107	3.3806	3.5111	5.4378
	0.8	0.27832	0.71224	1.4065	1.4573	2.0353	2.5448	3.2874	3.9747
15°	0.2	1.4771	4.2047	7.6520	11.735	17.616	22.176	25.929	27.784
	0.3	0.84448	2.5766	4.0341	6.8670	9.6492	12.033	12.905	15.990
	0.4	0.56767	1.9704	2.4504	5.0113	5.9861	7.1916	8.7116	10.487
	0.5	0.42308	1.6244	1.6905	3.8804	4.1409	4.7688	6.9347	7.1210
	0.6	0.34116	1.1543	1.5479	2.6693	3.3583	3.7214	4.8869	6.0336
	0.7	0.29410	0.86983	1.4756	1.9144	2.5778	3.4431	3.4633	5.4242
	0.8	0.26964	0.69703	1.4324	1.4387	2.0818	2.5218	3.3363	3.9280
30°	0.2	0.94477	4.4594	8.1448	11.392	20.412	21.694	27.243	30.041
	0.3	0.58204	2.6746	4.0343	6.6985	10.435	11.913	14.027	17.835
	0.4	0.42558	1.8747	2.5955	4.6852	6.3784	7.7910	8.7828	10.986
	0.5	0.33981	1.3660	2.0077	3.3906	4.6140	5.2916	6.3742	7.7987
	0.6	0.28828	1.0213	1.7435	2.4433	3.5926	4.0925	4.5835	6.4358
	0.7	0.25752	0.79473	1.6095	1.7986	2.7685	3.3155	3.6730	5.1650
	0.8	0.24179	0.65205	1.3643	1.5403	2.2095	2.4755	3.4856	3.8002
45°	0.2	0.61642	3.2701	9.3908	12.054	18.651	26.448	29.211	33.022
	0.3	0.37519	1.8930	5.3009	5.8233	10.267	13.448	16.599	17.673
	0.4	0.28260	1.3385	3.5875	3.6918	6.8931	8.2606	9.8056	11.420
	0.5	0.23615	1.0280	2.5902	2.7814	4.9621	5.9208	6.4826	8.2308
	0.6	0.21015	0.82020	2.0175	2.2211	3.7070	4.5097	4.8374	6.1884
	0.7	0.19603	0.67350	1.5765	1.9229	2.8417	3.3969	4.1161	4.7476
	0.8	0.19070	0.57563	1.2420	1.7579	2.2067	2.6668	3.4962	3.8837

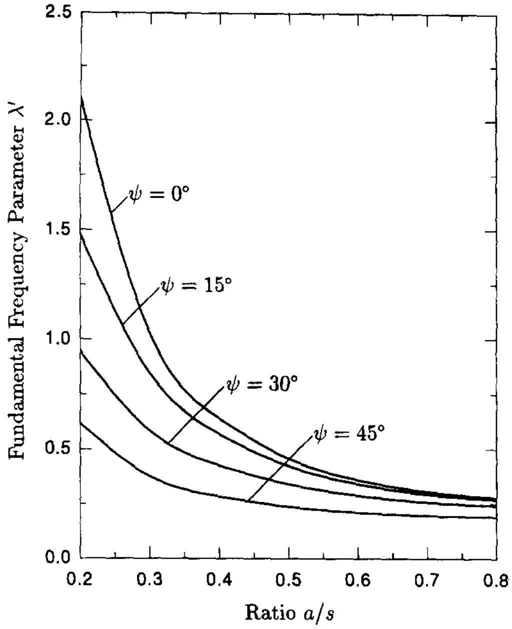

Fig. 2. Effects of angle of twist ψ on the fundamental frequency parameter λ' for a shallow conical shell with $v = 0.3$, $s/h = 1000$, $\theta_0 = 30°$ and $\theta_v = 15°$.

angle θ_v and base subtended angle θ_0 are respectively 15° and 30°. The angle of twist ψ varies from 0° (an untwisted shell) to 45°. The effects of a/s and initial twist ψ on ω can be observed from this table. The fundamental physical frequency ω decreases monotonically as a/s increases. The influence of initial twist on the fundamental λ' is illustrated in Fig. 2. The fundamental λ' decreases when ψ increases for a fixed a/s. This shows that the structural stiffness of the conical shell is reduced for a higher ψ. The significance of ψ on the stiffness (particularly the torsional stiffness) of a conical shell will be discussed at length shortly.

Figures 3–5 present the effects of ψ and a/s on the fundamental λ' for conical shells with $v = 0.3$, $s/h = 1000.0$, $\theta_0 = 30°$ and θ_v changing from 30°, 45° and 60°, respectively. It can be deduced that shells with a higher ψ reduce the fundamental λ'; this is valid for

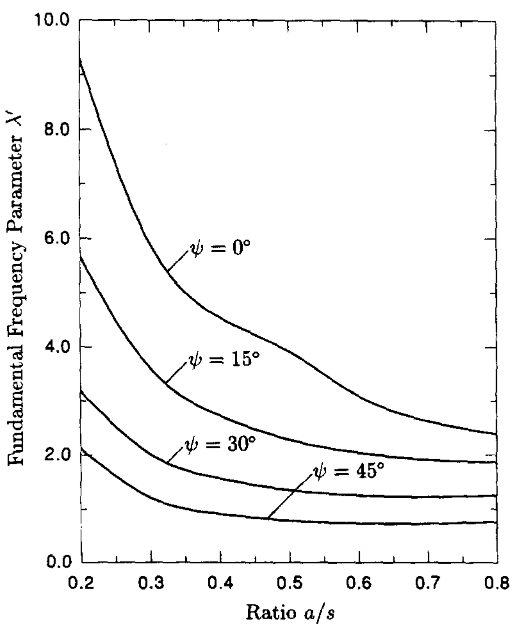

Fig. 3. Effects of angle of twist ψ on the fundamental frequency parameter λ' for a shallow conical shell with $v = 0.3$, $s/h = 1000$, $\theta_0 = 30°$ and $\theta_v = 30°$.

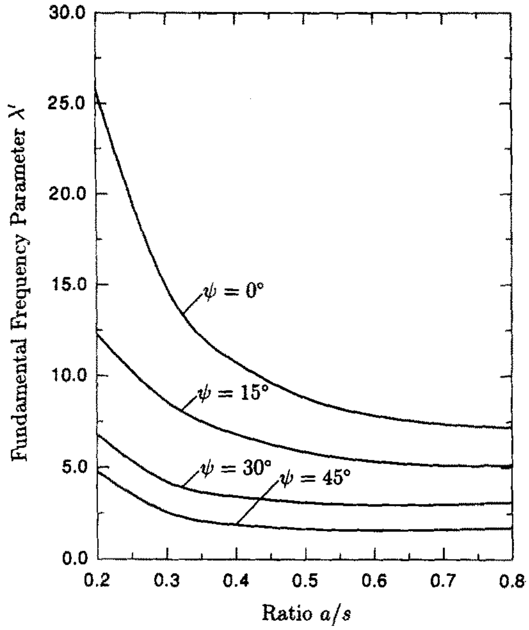

Fig. 4. Effects of angle of twist ψ on the fundamental frequency parameter λ' for a shallow conical shell with $\nu = 0.3$, $s/h = 1000$, $\theta_0 = 30°$ and $\theta_v = 45°$.

$\theta_v = 15°$, $30°$, $45°$ and $60°$. Comparing Figs 2–5, it is demonstrated that an increase in θ_v causes higher fundamental λ' for a fixed ψ.

Figure 6 illustrates the transverse vibration mode shapes of the pretwisted conical shell with $a/s = 0.2$, $\theta_v = 30°$ and $\theta_0 = 30°$. Although the in-plane vibration modes represented by U and V in eqns (10a,b) are not shown here, they also exist for shallow pretwisted conical shells. The shaded regions represent areas with negative vibration amplitude while the unshaded regions otherwise. However, the shaded and unshaded regions can be interchanged because only the vibration amplitude is of primary concern. The lines of demarcation in between the regions are the nodal line with zero displacement amplitude. From Fig. 6, it is evident that the fundamental vibration mode is the torsional mode (1-T) for $\psi = 0°$, $15°$, $30°$ and $45°$. It can also be seen that the first spanwise bending mode (1-SB)

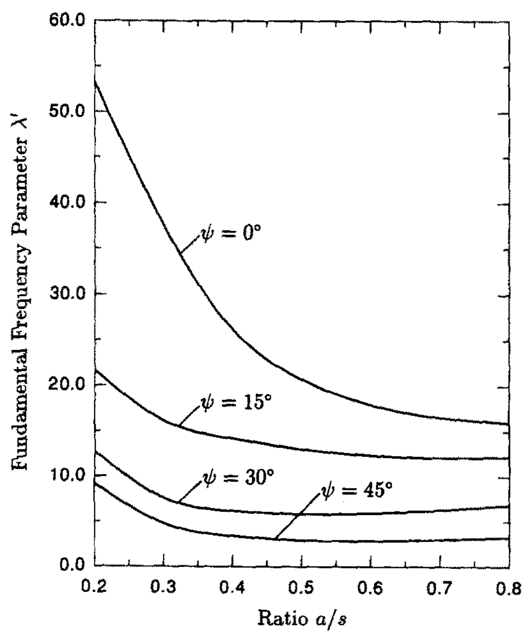

Fig. 5. Effects of angle of twist ψ on the fundamental frequency parameter λ' for a shallow conical shell with $\nu = 0.3$, $s/h = 1000$, $\theta_0 = 30°$ and $\theta_v = 60°$.

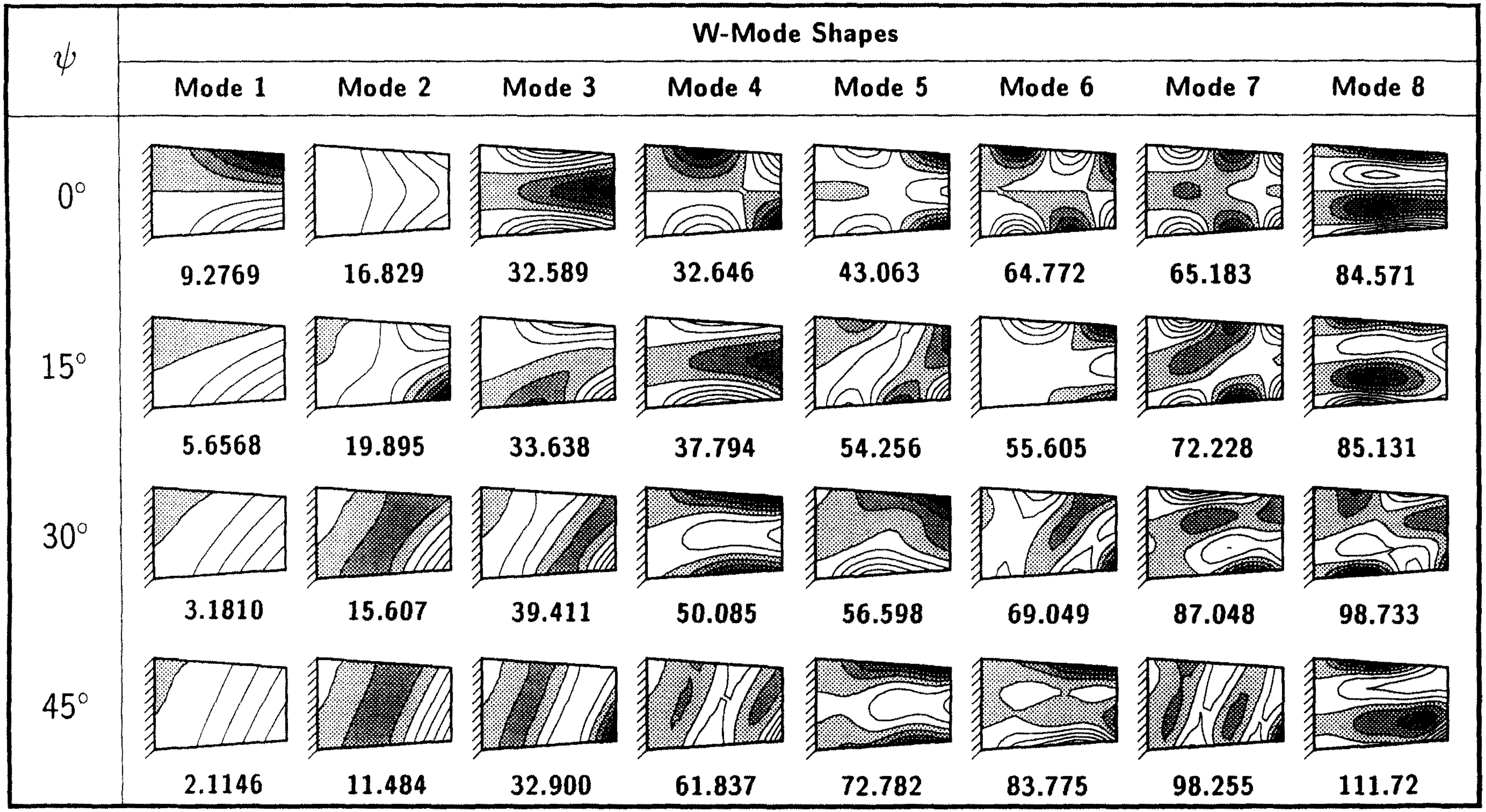

Fig. 6. Effects of angle of twist ψ on the transverse vibration mode shapes for a shallow conical shell with $\nu = 0.3$, $s/h = 1000$, $\theta_0 = 30°$, $\theta_v = 30°$ and $a/s = 0.2$.

occurs at mode 2 for $\psi = 0°$ which is the only mode without a nodal line. The subsequent modes in sequence are the 1-CB, 2-T, 1-SCB, 3-T, 2-SCB and 2-CB modes where CB stands for chordwise bending, T for torsional and SCB for combination of spanwise and chordwise bending modes.

It is interesting to note that λ' for the 1-T mode decreases as ψ increases. This observation seems contradictory to both physical intuition and previous publications (Chu, 1951; Rosen, 1980; Shield, 1982). However, this fact agrees well with the results reported recently by Tsuiji and Sueoka (1990). They have shown that the frequencies of torsional modes do not increase monotonically as the angle of twist increases. This implies that the torsional stiffness of a conical shell does not necessarily increase for a higher ψ, in contradiction to the experimental and computational results of previous researchers for beams (Chu, 1951; Rosen, 1980; Shield, 1982) and pretwisted plates (Leissa *et al.*, 1984; MacBain *et al.*, 1985; Kielb *et al.*, 1985). If we further observe eqns (5b,c), it is reasonable to point out that the torsional stiffness of a conical shell depends on the coupling effects of both the radius of curvature R_y and radius of twist R_{xy}. When R_y is absent (a pretwisted plate), a decrease in R_{xy} (thus an increase in ψ) results in higher torsional stiffness and frequency. As both of them (R_y and R_{xy}) are present, however, the above statement is no longer valid.

It should be noted that for $\psi = 0°$, all the vibration modes are symmetric with respect to the x-axis because the shell is untwisted. However, this symmetry of modes disappears when ψ increases because the geometry of shell is no longer symmetric. The higher modes for $\psi \neq 0°$ can be regarded as combinations of torsional, spanwise bending and chordwise bending modes and it is therefore impossible to classify the modes easily.

6. CONCLUSIONS

A new global energy procedure was proposed to examine the effects of initial twist on the vibration behaviour of shallow conical shells. The variational energy functional is minimized in accordance with the Ritz procedure. A set of *pb*-2 shape functions was introduced to represent the admissible displacement amplitude functions. These shape functions incorporate the piecewise boundary expressions each raised to an appropriate basic power to ensure the satisfaction of the geometric boundary conditions. The upper-bound convergent eigenvalues were carefully verified through a convergence study. It has been shown that the fundamental λ' decreases when ψ increases or θ_v decreases. The effect of a/s on the fundamental λ' can be seen in Table 2 or Figs 2–5. The fundamental physical frequency ω decreases monotonically as a/s increases. It has been shown that an increase in the angle of twist, though increases the torsional frequencies of a pretwisted beam or plate, does not ensure higher torsional stiffness for a conical shell. It has also been found that the symmetry of modes is absent when the angle of twist is non-zero.

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